

Emergent statistical-mechanical structure in the dynamics along the period-doubling route to chaos

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Abstract

We consider both the dynamics within and towards the supercycle attractors along the period-doubling route to chaos to analyze the development of a statistical-mechanical structure. In this structure the partition function consists of the sum of the attractor position distances known as supercycle diameters and the associated thermodynamic potential measures the rate of approach of trajectories to the attractor. The configurational weights for finite 2^N , and infinite $N \rightarrow \infty$, periods can be expressed as power laws or deformed exponentials. For finite period the structure is undeveloped in the sense that there is no true configurational degeneracy, but in the limit $N \rightarrow \infty$ this is realized together with the analog property of a Legendre transform linking entropies of two ensembles. We also study the partition functions for all N and the action of the Central Limit Theorem via a binomial approximation.

PACS 5.45.Ac Low-dimensional chaos

PACS 05.20.Gg Classical ensemble theory

PACS 05.45.Df Fractals

1 Introduction

For thermal systems formed by particles interacting via standard forces the limit of validity of equilibrium statistical mechanics is, trivially, non-equilibrium. Thermal systems constitute the normal realm of the Boltzmann-Gibbs (BG) formalism, but there are other types of systems for which it has been known for some time that they accept a statistical-mechanical description of the BG type. These are multifractals and chaotic nonlinear dynamical systems [1], among which one-dimensional unimodal iterated maps, represented

by the quadratic logistic map, are familiar model systems [2, 3] that exhibit such properties. The chaotic attractors generated by this class of maps have ergodic and mixing properties and not surprisingly they can be described by a thermodynamic formalism compatible with BG statistics [1]. But at the transition to chaos, the period-doubling accumulation point, the so-called Feigenbaum point, these two properties are lost and this suggests the possibility of exploring the limit of validity of the BG structure in a precise but simple enough setting.

Recently a comprehensive description has been given [4, 5] of the elaborate dynamics that takes place both inside and towards the Feigenbaum attractor. Amongst several conclusions, these studies established that the two types of dynamics are related to each other in a statistical-mechanical way, *i.e.* the dynamics at the attractor provides the ‘microscopic configurations’ in a partition function while the approach to the attractor is efficiently described by an entropy obtained from it. As we show below, this property conforms to q -deformations [4, 5, 6, 7], of the ordinary exponential weight of BG statistics. This novel statistical-mechanical feature arises in relation to a multifractal attractor with vanishing Lyapunov exponent. Here we explore in more detail this property with focus on how the statistical-mechanical structure develops along the period-doubling bifurcation cascade [2, 3], *i.e.* out of chaos.

Deformed exponentials appear in the studies of many physical systems. For instance, simulated velocity distributions of statistical-mechanical models resemble closely the so-called q -gaussian expression [8, 9], suggesting the occurrence of generalized statistical-mechanical structures under non-equilibrium conditions. Here, as an effort to provide a firm basis to a wider content discussion, we chose to study a nontrivial archetypal system under ergodicity and mixing failure and precisely determine its properties independently of any method that assumes a statistical-mechanical formalism. After that, the results obtained can be analyzed in relation to generalized entropy expressions or properties derived from them.

2 Brief recall of the dynamics within and towards the Feigenbaum attractor

The trajectories associated with the period-doubling route to chaos in unimodal maps exhibit elaborate dynamical properties that follow concerted patterns. At the period-doubling accumulation points, periodic attractors become multifractal before turning chaotic. At these points the Lyapunov exponent λ vanishes as it changes sign [2, 3]. There are two sets of properties associated with the attractors involved: those of the dynamics inside the attractors and those of the dynamics towards the attractors. These properties have been characterized in detail, the organization of trajectories and also that of the sensitivity to initial conditions at the Feigenbaum attractor are described in Ref. [4], while the features of the rate of approach of an ensemble of trajectories to this attractor has been explained in Ref. [5].

We recall some of the basic features of the bifurcation forks that form the period-doubling cascade sequence in unimodal maps, often illustrated by the logistic map $f_\mu(x) = 1 - \mu x^2$, $-1 \leq x \leq 1$, $0 \leq \mu \leq 2$ [2, 3]. The knowledge of the dynamics towards a particular

family of periodic attractors, the so-called superstable attractors [2, 3], facilitates the understanding of the rate of approach of trajectories to the Feigenbaum attractor, located at $\mu = \mu_\infty = 1.401155189092\dots$, and highlights the source of the discrete scale invariant property of this rate [5]. The family of trajectories associated with these attractors - also called supercycles - of periods 2^N , $N = 1, 2, 3, \dots$, are located along the bifurcation forks. The positions (or phases) of the 2^N -attractor are given by $x_j = f_{\bar{\mu}_N}^{(j)}(0)$, $j = 1, 2, \dots, 2^N$. Associated with the 2^N -attractor at $\mu = \bar{\mu}_N$ there is a $(2^N - 1)$ -repellor consisting of $2^N - 1$ positions y_k , $k = 0, 1, 2, \dots, 2^N - 1$. These positions are the unstable solutions, $\left| df_{\bar{\mu}_N}^{(2^{n-1})}(y)/dy \right| > 1$, of $y = f_{\bar{\mu}_N}^{(2^{n-1})}(y)$, $n = 1, 2, \dots, N$. The first, $n = 1$, originates at the initial period-doubling bifurcation, the next two, $n = 2$, start at the second bifurcation, and so on, with the last group of 2^{N-1} , $n = N$, setting out from the $N - 1$ bifurcation. The diameters $d_{N,m}$ are defined as $d_{N,m} \equiv x_m - f_{\bar{\mu}_N}^{(2^{N-1})}(x_m)$ [2, 3].

Central to our understanding of the dynamical properties of unimodal maps is the following in-depth property: Time evolution at μ_∞ from $\tau = 0$ up to $\tau \rightarrow \infty$ traces the period-doubling cascade progression from $\mu = 0$ up to μ_∞ . There is an underlying quantitative relationship between the two developments. Specifically, the trajectory inside the Feigenbaum attractor with initial condition $x_0 = 0$, the 2^∞ -supercycle orbit, takes positions x_τ such that the distances between appropriate pairs of them reproduce the diameters $d_{N,m}$ defined from the supercycle orbits with $\bar{\mu}_N < \mu_\infty$. See Fig. 1 in Ref. [5]. This property has been basic in obtaining rigorous results for the sensitivity to initial conditions for the Feigenbaum attractor [4], and for the dynamics of approach to this attractor [5]. Other families of periodic attractors share most of the properties of supercycles.

The organization of the total set of trajectories as generated by all possible initial conditions as they flow towards a period 2^N attractor has been determined in detail [5, 10]. It was found that the paths taken by the full set of trajectories in their way to the supercycle attractors (or to their complementary repellors) are exceptionally structured. The dynamics associated to families of trajectories always displays a characteristically concerted order in which positions are visited, and this is reflected in the dynamics of the supercycles of periods 2^N via the successive formation of gaps in phase space (the interval $-1 \leq x \leq 1$) that finally give rise to the attractor and repellor multifractal sets. To observe explicitly this process an ensemble of initial conditions x_0 distributed uniformly across phase space was considered and their positions were recorded at subsequent times [5, 10]. This set of gaps develops in time beginning with the largest one associated with the first repellor position, then followed by a set of two gaps associated with the next two repellor positions, next a set of four gaps associated with the four next repellor positions, and so forth. The gaps that form consecutively all have the same width in the logarithmic scales [5], and therefore their actual widths decrease as a power law, the same power law followed, for instance, by the position sequence $x_\tau = \alpha^{-N}$, $\tau = 2^N$, $N = 0, 1, 2, \dots$, for the trajectory inside the attractor starting at $x_0 = 0$ (and where $\alpha \simeq 2.50291$ is the absolute value of Feigenbaum's universal constant). The locations of this specific family of consecutive gaps advance monotonically toward the sparsest region of the multifractal attractor located at $x = 0$. See Refs. [4, 5, 10].

3 Sums of diameters as partition functions

The rate of convergence W_t of an ensemble of trajectories towards any attractor/repellor pair along the period-doubling cascade is a convenient single-time quantity that has a straightforward definition and is practical to implement numerically. A partition of phase space is made of N_b equally-sized boxes or bins and a uniform distribution of N_c initial conditions is placed along the interval $-1 \leq x \leq 1$. The ratio N_c/N_b can be adjusted to achieve optimal numerical results [5]. The quantity of interest is the number of boxes W_t that contain trajectories at time t . This rate has been determined for the supercycles $\bar{\mu}_N$, $N = 1, 2, 3, \dots$, and its accumulation point μ_∞ [5]. See Fig. 19 in that reference where W_t is shown in logarithmic scales for the first five supercycles of periods 2^1 to 2^5 where we can observe the following features: In all cases W_t shows a similar initial and nearly constant plateau $W_t \simeq \Delta$, $1 \leq t \leq t_0$, $t_0 = O(1)$, and a final well-defined decay to zero. As it can be observed in the left panel of Fig. 19 in [5], the duration of the final decay grows approximately proportionally to the period 2^N of the supercycle. There is an intermediate slow decay of W_t that develops as N increases with duration also just about proportional to 2^N . For the shortest period 2^1 , there is no intermediate feature in W_t ; this appears first for period 2^2 as a single dip and expands with one undulation every time N increases by one unit. The expanding intermediate regime exhibits the development of a power-law decay with logarithmic oscillations (characteristic of discrete scale invariance). In the limit $N \rightarrow \infty$ the rate takes the form $W_t \simeq \Delta h(\ln \tau / \ln 2) \tau^{-B}$, $\tau = t - t_0$, where $h(x)$ is a periodic function with $h(1) = 1$ and $B \simeq 0.8001$ [5].

The rate W_t , at the values of time for period doubling, $\tau = 2^n$, $n = 1, 2, 3, \dots < N$, can be obtained quantitatively from the supercycle diameters $d_{n,m}$. Specifically,

$$Z_\tau \equiv \frac{W_t}{\Delta} = \sum_{m=0}^{2^{n-1}-1} d_{n,m} \quad (1)$$

In the above expression, $\tau = t - t_0 = 2^{n-1}$, $n = 1, 2, 3, \dots < N$. Eq. (1) expresses the numerical procedure followed in [11] to evaluate the exponent B but it also suggests a statistical-mechanical structure if Z_τ is identified as a partition function where the diameters $d_{n,m}$ play the role of configurational terms [5]. The diameters $d_{N,m}$ scale with N for m fixed as $d_{N,m} \simeq \alpha_y^{-(N-1)}$, N large, where the α_y are universal constants obtained from the finite discontinuities of Feigenbaum's trajectory scaling function $\sigma(y) = \lim_{N \rightarrow \infty} (d_{N+1,m}/d_{N,m})$, $y = \lim_{N \rightarrow \infty} (m/2^N)$ [2, 5]. The largest two discontinuities of $\sigma(y)$ correspond to the sparsest and denser regions of the multifractal attractor at μ_∞ , for which we have, respectively, $d_{N,0} \simeq \alpha^{-(N-1)}$ and $d_{N,1} \simeq \alpha^{-2(N-1)}$ ($d_{1,0} = 1$).

4 A closer analysis of the partition functions for the supercycles

We proceed now to study in more detail the diameters $d_{N,m}$ so that we can evaluate the soundness of their association with configurational terms in a partition function. With this in mind we determined their values for the supercycles of periods 2^N from $N = 1$

to $N = 12$, that is, starting with the case of a single diameter $d_{1,0} = 1$ and following successively up to a set of 2048 diameters $d_{12,m}$, $m = 0, 1, \dots, 2^{12} - 1$. This task required the precise evaluation of the control parameter values $\bar{\mu}_N$, $N = 1, \dots, 12$.

In Fig. 1 we show the lengths of these sets when arranged with decreasing values, namely, we present the $d_{N,m}$ as a function of their rank r , the size-rank distributions, in logarithmic scales, as it is often done for these type of distributions that exhibit frequently power law behavior. We observe in Fig. 1 that the distributions have a downhill terraced (or multiple-plateau) structure, the diameters form well-defined size groups and these sizes decrease on average a fixed amount (in the logarithmic scales shown) equal to $\log_{10} \alpha \simeq 0.39844$ from group to group. This amount reflects the well-known [2, 3] power-law scaling of diameter sizes via the universal constant α . Their size-rank distributions satisfy a piecewise Zipf-like law. For example, for the largest diameter we have $d_{N,0}/d_{N+1,0} \simeq \alpha^1$, whereas for the smallest we have $d_{N,1}/d_{N+1,1} \simeq \alpha^2$. Therefore $d_{N,0} \simeq \alpha^{-N}$ and $d_{N,1} \simeq \alpha^{-2N}$.

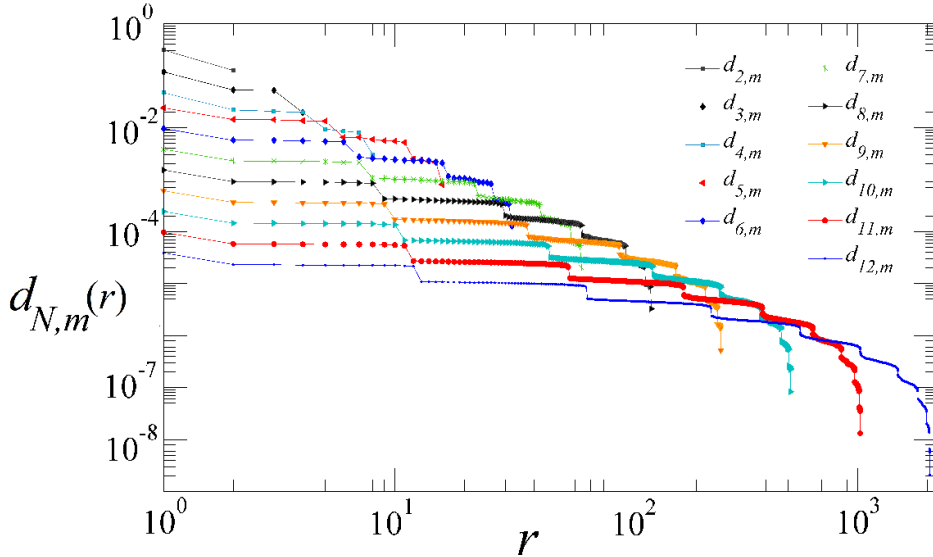


Figure 1: Length-rank distributions of diameters $d_{N,m}$ for the first 12 supercycles in logarithmic scales. The distributions have a downhill terraced or multiple-plateau structure and the diameters form well-defined length groups. Their values for fixed period 2^N decrease on average as $\log_{10} \alpha \simeq 0.39844$ from group to group with N fixed or from N to $N + 1$ for the same kind of group.

We observe clearly in Figs. 1 and 3 that the diameter lengths within each group are not equal, so that there is no degeneracy in them. However the differences in lengths within groups diminishes rapidly as N increases. There are two groups with only one member, the largest and the shortest diameters, and the numbers within each group grow monotonically from each end towards the middle-sized length group. The numbers of diameters forming these groups can be neatly arranged into a Pascal Triangle (see Fig. 2), and therefore we anticipate the action of the Central Limit Theorem, in a form reminiscent of the De Moivre-Laplace theorem, so that in the limit $N \rightarrow \infty$, the middle-sized-length group of diameters dominates the partition function Z_τ and a situation similar to the saddle-point approximation occurs. Also, in the limit $N \rightarrow \infty$ the lengths of the dominant group (as

well as those of all other groups of diameters with smaller lengths) become closer in size (see the trend in Fig. 1), so that in the limit $N \rightarrow \infty$ there appears a true degeneracy in the dominant partition function configurations that gives the statistical-mechanical structure the required characteristics for ensemble equivalence and the Legendre transform property central to statistical mechanics.

The above facts and understanding allow us to be more precise and we denote now the diameters as $d_{N,l,i}$ where the subindexes l and i provide more specific information than the former subindex m . Subindex $l = 0, 1, \dots, N-1$ designates the group terrace (as in Fig. 1), with decreasing size as l increases, and the subindex $i = 1, 2, \dots, \binom{N-1}{l}$ identifies the individual diameter within the group, again with decreasing size as i increases. The diameters $d_{N,l,i}$ are written as $d_{N,l,i} = A_{N,l,i} \alpha^{-(N-1-l)} \alpha^{-2l}$ where the scaling factors $\alpha^{-(N-1-l)}$ and α^{-2l} give the size of the group terrace (see Fig. 2) and the amplitude $A_{N,l,i}$ fixes the value of the individual diameter. (Due to the De Moivre-Laplace theorem, in the limit $N \rightarrow \infty$ the amplitude $A_{N,N/2,i}$ can be obtained from an inverse complementary error function, but we do not expand on this here). The scaling factor $\alpha^{-(N-1+l)} = \alpha^{-(N-1-l)} \alpha^{-2l}$ can be rewritten exactly as a q -exponential, defined as $\exp_q(x) \equiv [1 + (1-q)x]^{1/(1-q)}$, via use of the identity $\alpha^{-(N-1+l)} \equiv (1 + \epsilon_l)^{-\ln \alpha / \ln 2}$, $\epsilon_l = 2^{N-1+l} - 1$. That is, $d_{N,l,i} \simeq A_{N,l,i} \exp_q(-\beta \epsilon_l)$, where, $q = 1 + \beta^{-1}$, $\beta = \ln \alpha / \ln 2$. Similarly, $Z_\tau \simeq \tau^{-B}$ can be expressed as $Z_\tau \simeq \exp_Q(-B\epsilon)$, where $Q = 1 + B^{-1}$ and $\epsilon = \tau - 1 = 2^{N-1} - 1$. Therefore, taking the above into account in Eq. (1) we have

$$\exp_Q(-B\epsilon) \simeq \sum_{l,i} A_{N,l,i} \exp_q(-\beta \epsilon_l), \quad (2)$$

Eq. (2) resembles a basic statistical-mechanical expression except for the presence of the amplitudes $A_{N,l,i}$ and the fact that q -deformed exponential weights appear in place of ordinary exponential weights (that are recovered when $Q = q = 1$).

To explore further we use a binomial approximation for Z_τ [5]. That is, we adopt the approximation of considering the diameter lengths in each group to actually have equal length ($A_{N,l,i} = 1$) and assume that this common lengths are given by the binomial combination of the scale factors of those diameters that converge to the most crowded and most sparse regions of the multifractal attractor. Namely, the 2^{N-1} diameters at the N -th supercycle have lengths equal to $\alpha^{-(N-1-l)} \alpha^{-2l}$ and occur with multiplicities $\binom{N-1}{l}$ where $l = 0, 1, \dots, N-1$. See Fig. 2. The imposed degeneracy within groups in the diameter lengths complete the Pascal Triangle structure across the bifurcation cascade. This feature significantly simplifies the evaluation of the partition function in Eq. (1) and directly yields

$$Z_\tau = \sum_{l=0}^{N-1} \binom{N-1}{l} \alpha^{-(N-1-l)} \alpha^{-2l} = (\alpha^{-1} + \alpha^{-2})^{N-1} \quad (3)$$

where $\tau = 2^{N-1}$. We obtain $B = 0.8386$, and $Q = 2.1924$, a surprisingly good approximation when compared to the numerical estimates $B = 0.8001$ and $Q = 2.2498$ of the exact values [5]. Eq. (2) reads now

$$\exp_Q(-\beta F) = \sum_{l=0}^{N-1} \Omega(N-1, l) \exp_q(-\beta \epsilon_l), \quad (4)$$

where $F/\epsilon = (1-q)/(1-Q)$, and $\Omega(N-1, l) = \binom{N-1}{l}$, $\alpha^{-(N-1-l)}\alpha^{-2l} = 2^{-(N-1+l)(\ln \alpha / \ln 2)} = \exp_q(-\beta \epsilon_l)$. In the language of thermal systems Eq. (4) reads as follows: There are $N-1$ degrees of freedom that generate 2^{N-1} configurations, and these occupy N energy levels with degeneracies $\Omega(N-1, l)$, $l = 0, 1, \dots, N-1$. Under the binomial approximation the energies of the 2^{N-1} configurations become confined into the energy values $\epsilon_l = 2^{(N-1+l)}$, $l = 0, 1, \dots, N-1$. In the generalized canonical partition function all the q -exponential weights acquire a fixed inverse temperature $\beta = \ln \alpha / \ln 2$. When we extend the study of the quadratic map to the infinite family of unimodal maps with extremum of nonlinearity $1 < z < \infty$ the inverse temperature β can be varied continuously, as the universal constant $\alpha(z)$ varies monotonically with z [12].

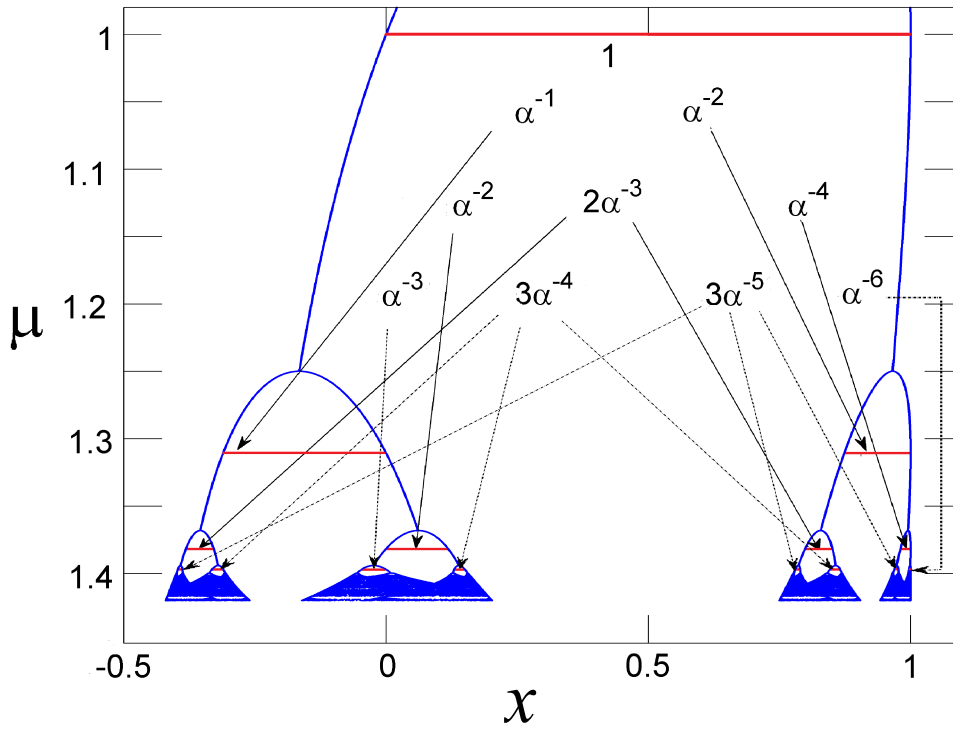


Figure 2: Sector of the bifurcation tree for the logistic map $f_\mu(x)$ that shows the formation of a Pascal Triangle of diameter lengths according to the binomial approximation explained in the text, where $\alpha \simeq 2.50291$ is the absolute value of Feigenbaum's universal constant.

5 A limiting statistical-mechanical structure for the dynamics at the Feigenbaum point

According to our scheme, for finite N (the supercycle of period 2^N at $\bar{\mu}_N$) we can form $N-1$ partition functions Z_τ , $\tau = 2^{n-1}$, $n = 1, 2, 3, \dots, N-1$. The number of terms

in these partition functions range from a single term, $d_{1,0}$, to 2^{N-1} terms, $d_{N,m}$, $m = 0, 1, 2, \dots, 2^{N-1} - 1$. As explained, for uniform distributions of initial conditions $-1 \leq x_0 \leq 1$ at $\mu = \bar{\mu}_N$, the partition functions Z_τ measure the fraction of ensemble trajectories still away from the attractor at times $\tau = 2^{n-1}$, $n = 1, 2, 3, \dots, N - 1$. These times coincide with the sequential process of phase-space gap formation by the trajectories [5]. The gaps correspond to the intervals in $-1 \leq x \leq 1$ located between the bifurcation forks in the period-doubling cascade, when $\mu = \bar{\mu}_N$, that is, the gap intervals are placed between consecutive diameters. As N grows new smaller gaps proliferate while the new diameters grow in number and each of them decreases in value. See Fig. 3 where the numbers of diameters are shown for each group formed for the case of the 12th supercycle. The number of groups into which the diameters distribute increases as it does the number of diameters within each group. As we have indicated these increments obey the entries in the Pascal Triangle generated by a binomial. Although the diameters within each group are never equal their differences decrease rapidly. The dominant term in Z_τ is that associated with $\Omega(N-1, (N-1)/2)$, N odd, and in the limit $N \rightarrow \infty$ we have that $Z_\infty = \Omega(N \rightarrow \infty, l = N/2 \rightarrow \infty)$. We interpret this last equality as ensemble equivalence in the thermodynamic limit (here $N \rightarrow \infty$ is the attractor at the transition to chaos).

It is more convenient to describe the ensemble equivalence in terms of the binomial approximation of the partition function Z_τ given by Eqs. (3) and (4), where $\Omega(N-1, l)$ plays the role of a ‘microcanonical’ partition function representing the system configurations with fixed diameter length $\alpha^{-(N-1-l)}\alpha^{-2l}$ and Z_τ stands for the ‘canonical’ partition function that is formed by weighting the degenerate configurations $\Omega(N-1, l)$ for each length group by the factor $\alpha^{-(N-1+l)} \equiv \exp_q(-\beta\epsilon_l)$. According to the De Moivre-Laplace early form of the Central Limit Theorem the growth of N drives the binomial distribution towards a Gaussian distribution

$$\delta^N P_{l, N-l} \simeq \frac{1}{\sqrt{2\pi N \rho \sigma}} \exp\left(-\frac{x^2}{2N \rho \sigma}\right), \quad (5)$$

where $\delta = \alpha^{-1} + \alpha^{-2}$, $\rho \sim \alpha^{-2}$, $\sigma \sim \alpha^{-1}$ and $x = l - N\rho$. For large N the midpoint terms in the expansion of the binomial dominate, $Z_\tau = (\alpha^{-1} + \alpha^{-2})^{N-1} \simeq \binom{N}{N/2} \alpha^{-3N/2} \sim 2^N \alpha^{-3N/2}$, and in the limit $N \rightarrow \infty$ we have that $Z_\infty = \Omega(N \rightarrow \infty, l = N/2 \rightarrow \infty)$. For N fixed the ‘energies’ ϵ_l range from $2^{N-1} - 1$ to $2^{2(N-1)} - 1$. For large N the ‘energy’ that corresponds to the ‘microcanonical’ partition function that becomes the dominant term in Z_τ is $\epsilon_{N/2} \simeq 2^{3/2N}$.

A crossover to ordinary BG type statistics takes place when $\mu \gtrsim \mu_\infty$ and the attractor becomes chaotic. For $\Delta\mu \equiv \mu - \mu_\infty > 0$ the attractors are made up of 2^N , $N = 1, 2, 3, \dots$, bands, with N larger for $\Delta\mu$ smaller, while the Lyapunov exponent scales as $\lambda \sim 2^{-N}$. The trajectories consist of an interband periodic motion of period 2^N and an intraband chaotic motion. As explained in Ref. [5] the consideration of backward iterations in unimodal maps, together with the expansion of separation of trajectories when $\lambda > 0$, can be invoked to write a partition function similar to that in Eq. 2 but now with ordinary exponentials as configurational weights.

As it is well known [1], the so-called thermodynamic formalism for the description of the geometric properties of multifractal sets is built around a statistical-mechanical framework of the BG type. The partition function formulated to study multifractal properties,

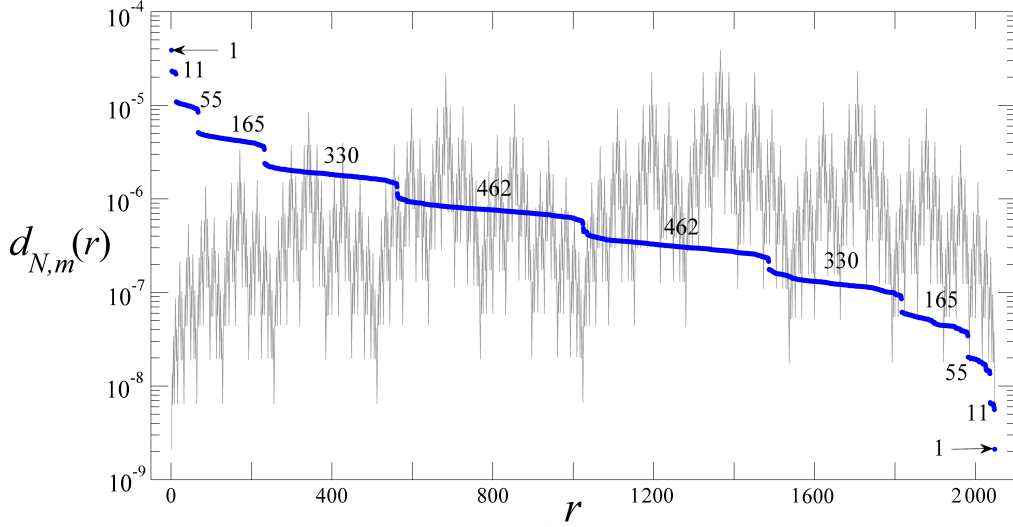


Figure 3: Length-rank distribution for the 2048 diameters of the 12th supercycle in semi-logarithmic scales. The figures indicate the number of diameters in each group. As N grows the length-rank distributions approach the binomial size-rank distribution and the De Moivre-Laplace theorem applies at the transition to chaos. See text. In the background we show the same diameters before sorting them out as a function of m

like the spectrum of singularities $f(\tilde{\alpha})$, is written as

$$Z(\tilde{\tau}, \tilde{\beta}) = \sum_m^M p_m^{\tilde{\tau}} l_m^{\tilde{\beta}}, \quad (6)$$

where the l_m in one-dimensional systems are M disjoint interval lengths that cover the multifractal set and the p_m are probabilities given to these intervals. The standard practice consists of demanding that $Z(\tilde{\tau}, \tilde{\beta})$ neither vanishes nor diverges in the limit $l_m \rightarrow 0$ for all m (notice that in this limit $M \rightarrow \infty$). Under this condition the exponents $\tilde{\tau}$ and $\tilde{\beta}$ define a function $\tilde{\tau}(\tilde{\beta})$ from which $f(\tilde{\alpha})$ is obtained via Legendre transformation [1]. When the multifractal is an attractor its elements are ordered dynamically, and for the Feigenbaum attractor the trajectory with initial condition $x_0 = 0$ generates in succession the positions that form the diameters, generating the entire set of diameters $d_{N,m}$, $m = 0, 1, 2, \dots, 2^{(N-1)} - 1$, $N = 1, 2, \dots$. Because the diameters cover the attractor it is natural to choose the covering lengths at stage N to be $l_m^{(N)} = d_{N,m}$ and to assign to each of them the same probability $p_m^{(N)} = (1/2)^{N-1}$, and the condition $Z(\tilde{\tau}, \tilde{\beta}) = 1$ reproduces Eq. (1) when $p_m^{(N)} = \tau^{-1} = (1/2)^{N-1}$ with $\tilde{\tau} = -B$ and $\tilde{\beta} = 1$.

6 Summary and discussion

The items we studied are the following: i) The partition function we considered is the sum of attractor position distances (the so-called diameters of the supercycles [2, 3]) for each period 2^N along the bifurcation cascade that leads to the transition to chaos. ii) For

uniformly-distributed sets of initial conditions x_0 the partition function is equal to the number of bins that still contain trajectories en route to the attractor at time $\tau = 2^n$, $n = 1, 2, 3, \dots \lesssim N$, where the supercycle period is $\tau = 2^N$, $N > 1$. iii) For N fixed the values of the diameters distribute into well-defined groups with a size-rank structure that develops into a power law as N increases. These groups can be arranged into a Pascal Triangle when considering all N up to $N \rightarrow \infty$, but the diameters within each group are not equal. Nevertheless, their differences diminish rapidly as N increases, so that a binomial *approximation* can be introduced such that the diameters within each group are considered equal for all N . iv) In the limit $N \rightarrow \infty$ the diameter-group degeneracy imparts the partition function the required structure to observe ensemble equivalence, and other familiar features of statistical mechanics, even though the configurational weights are not exponential. v) The visible or ‘macroscopic’ manifestation of the statistical-mechanical structure, the emergence of a power law with log-periodic modulation associated with the rate of approach of trajectories towards the Feigenbaum attractor, is linked to the sequential process of phase-space gap formation. vi) Beyond the transition to chaos, when the attractors become sets of chaotic bands, the configurational weights are converted into ordinary exponentials and the usual BG form is recovered.

The main advance presented here with respect to Ref. [5] is the determination of the terrace structure displayed by the diameters for finite N shown in Figs. 1 and 3. This fact allowed us to write the partition function in Eq. 1 explicitly as Eq. 2. Therefore we were able to study how the lack of configurational degeneracy gradually disappears as $N \rightarrow \infty$ leading to ensemble equivalence.

Chaotic dynamics in nonlinear systems accepts statistical-mechanical descriptions [1]. Unimodal maps, usually represented by the logistic map, offer a simple but nontrivial model system in which to explore the development of such a statistical-mechanical structure, to examine the gradual fulfilment of basic elements and eventually the display of the full ordinary features of the BG formalism. A unimodal map is a well-defined and controllable numerical laboratory for the observation of the limit of validity of the BG formalism when the ergodic and mixing properties of chaotic dynamics break down. As it has long been known unimodal maps display two bifurcation cascades that take place in opposite directions in control parameter space, one for $\mu < \mu_\infty$ when periodic attractors double their periods, and the other for $\mu > \mu_\infty$ when chaotic-band attractors split doubling their number of bands. The two cascades meet at $\mu = \mu_\infty$. Infinitely many reproductions of these inverse cascades appear within the windows of periodicity that interrupt the chaotic-band attractors for $\mu > \mu_\infty$ [2, 3].

As we have mentioned, the ergodic and mixing trajectories of chaotic-band attractors conform to a statistical mechanical-structure of the BG type [1]. We have described that the positions of periodic attractors can be used to define partition functions and that these capture information on the dynamics towards the attractors [5]. However, as we have explained, these partition functions lack some standard properties required in a thermodynamic formalism, such as the degeneracy of configurational states that manifests as ensemble equivalence and the correspondence of their respective thermodynamic potentials in the thermodynamic limit (that in the unimodal map model is the limit $N \rightarrow \infty$ of infinite period). For finite N the configurational terms (diameters $d_{N,m}$) separate into well-defined magnitude (length) groups but they are not equal within each group. These

groups of diameters are the prototypes of ‘microcanonical’ ensembles while the consideration of all groups, all diameters for a given supercycle of period 2^N is the candidate version of the ‘canonical’ ensemble. As we have seen, when $N \rightarrow \infty$ the diameters seem to fulfill a binomial approximation such that the (vanishing) lengths within the dominant diameter groups (with divergent numbers) become equal and the De Moivre-Laplace theorem establishes the equivalence between the ‘microcanonical’ and ‘canonical’ ensembles. The binomial approximation we presented for finite N allows for a conventional interpretation in the language of thermal systems.

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